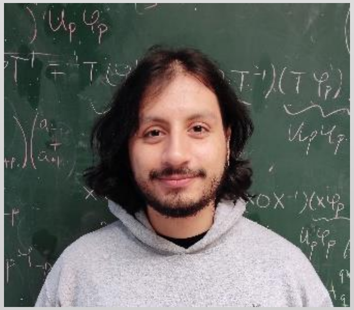
Fragility of spectral flow in topological insulators

Matthew S. Foster

December 12, 2023

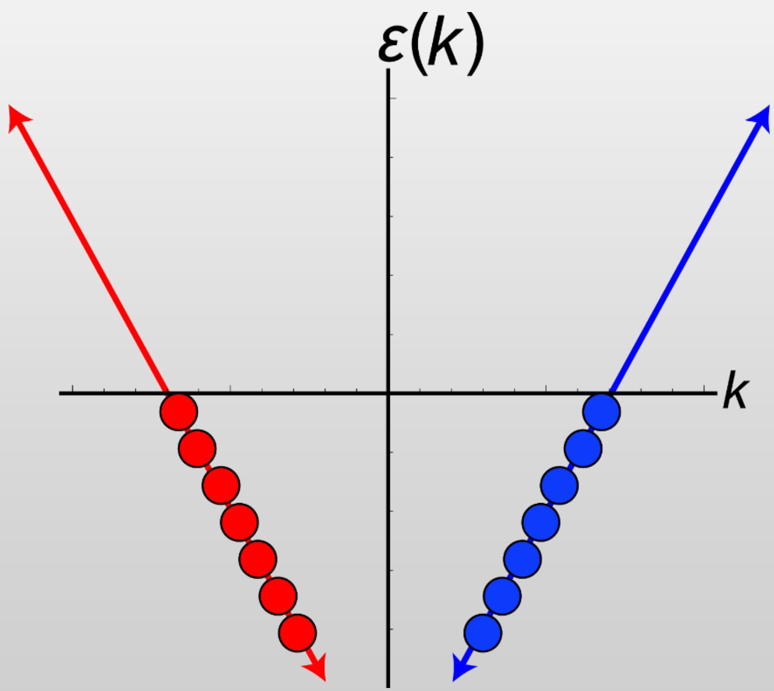
    

A. Altland, P. Brouwer, J. Dieplinger, M. S. Foster, M. Moreno-Gonzalez, and L. Trifunovic
arXiv:2308.12931

Chiral anomaly and spectral flow

- **Massless Dirac Hamiltonian in *d* = 1:** $\hat{h} = -i v_F \, \hat{\sigma}^3 \, \dfrac{d}{dx}$

- **This is a 2-band semimetal**



Chiral anomaly and spectral flow

- **Massless** **Dirac Hamiltonian in *d* = 1:** $\hat{h} = -i v_F \, \hat{\sigma}^3 \, \frac{d}{dx}$

- **Couple 1+1-D Dirac to electromagnetism: Axial anomaly**

1. **Electric "2-current" is conserved** $\partial_t \rho + \partial_x J = 0$

2. **Axial (swap) 2-current is not**

$$\partial_t \left(\frac{J}{v_F} \right) + \partial_x \left(v_F \, \rho \right) = \frac{2e^2}{h} E$$

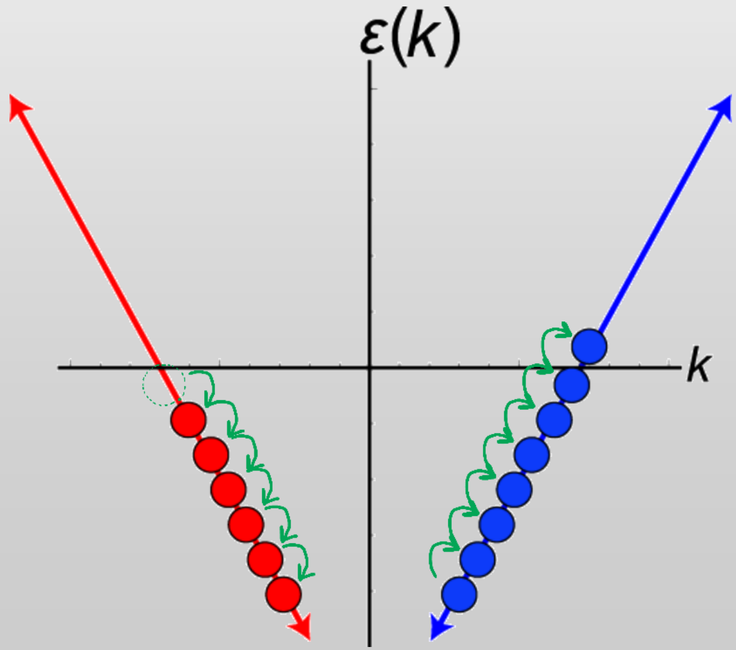
Schwinger 1962

Chiral anomaly and spectral flow

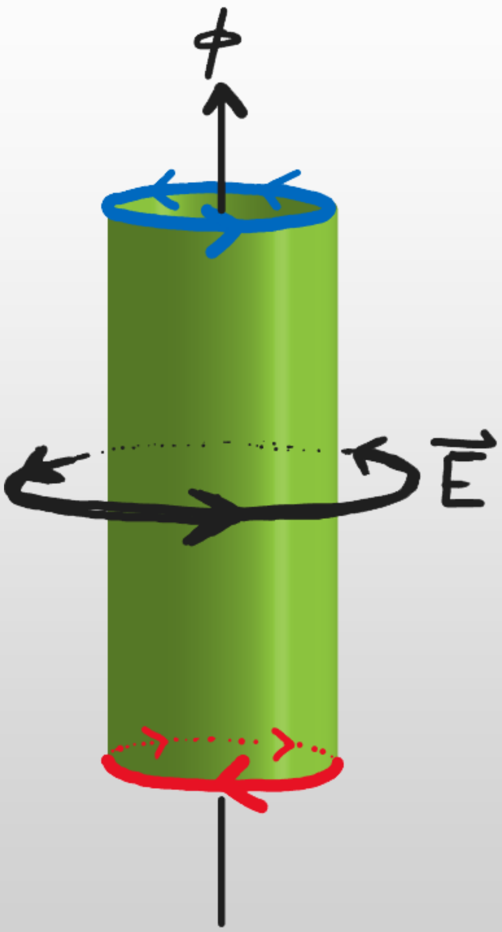
- **Axial current non-conservation: Material realizations**

$$\frac{1}{v_F}\frac{dJ}{dt} = e\frac{d}{dt}\left(n_R - n_L\right) = \frac{2e^2}{h}E$$

1. **Newton's 2[nd] law, 1D wire:**
   Left-movers convert to right-movers (through the bottom of the band, not shown)



What if we could physically separate right- and left-movers?



Chiral anomaly and spectral flow

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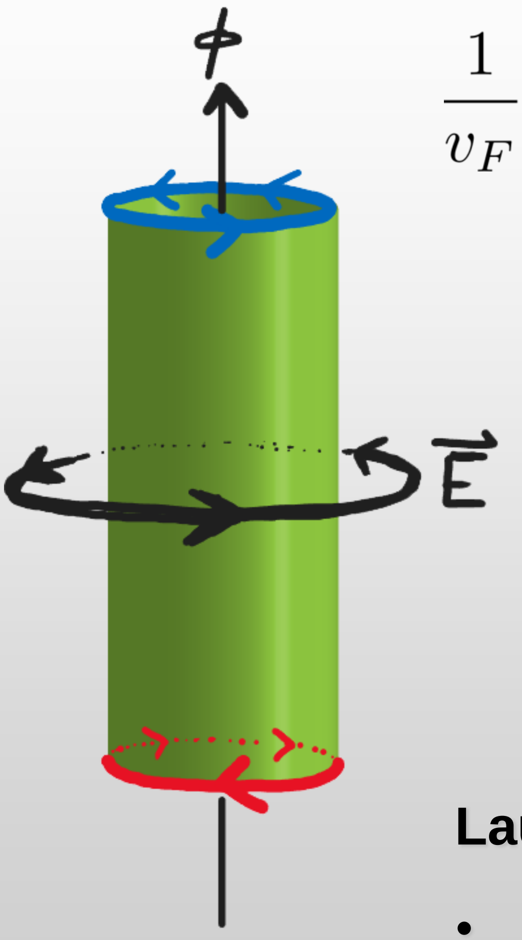
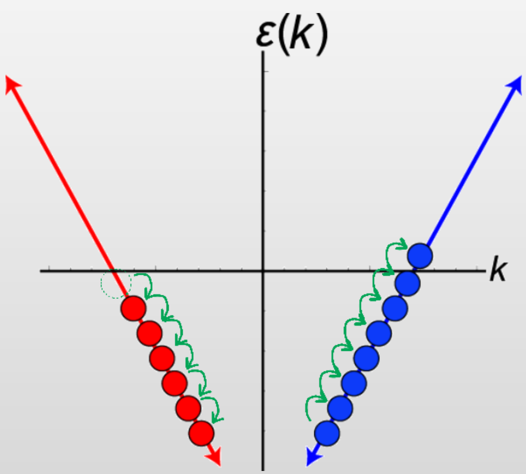


$$\frac{1}{v_F}\frac{dJ}{dt} = e\frac{d}{dt}\left(n_R - n_L\right) = \frac{2e^2}{h}E$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\Delta N_R - \Delta N_L = \frac{2e}{hc}\Delta\phi$$



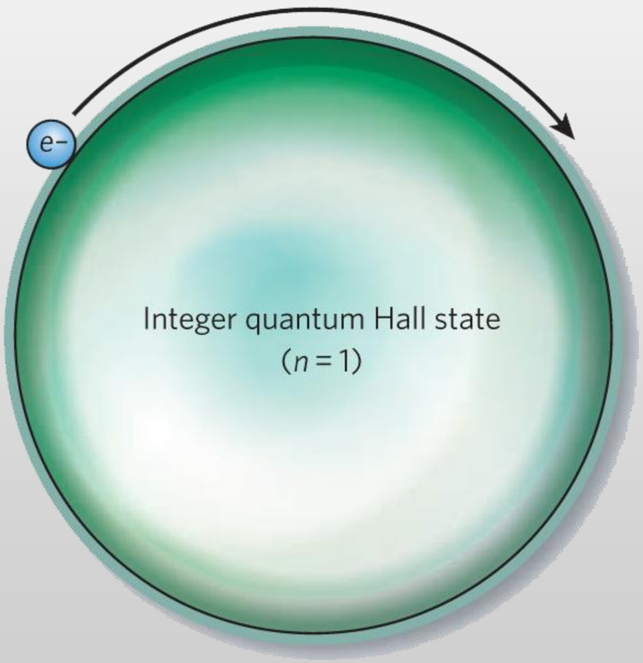
Laughlin 1981, Halperin 1982:

- **Pump from right- to left- edges:**
- **“Spectral Flow Principle” for topological matter!**

Anderson Localization is ubiquitous in 1,2-D…Topological exceptions

1. **1D chiral edge states of a quantum Hall droplet**

$$\hat{h} = -iv_F \frac{d}{dx}$$



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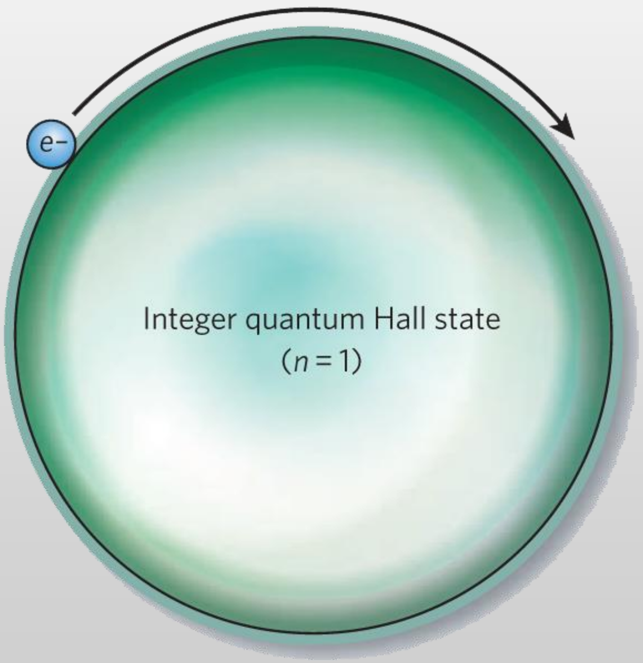
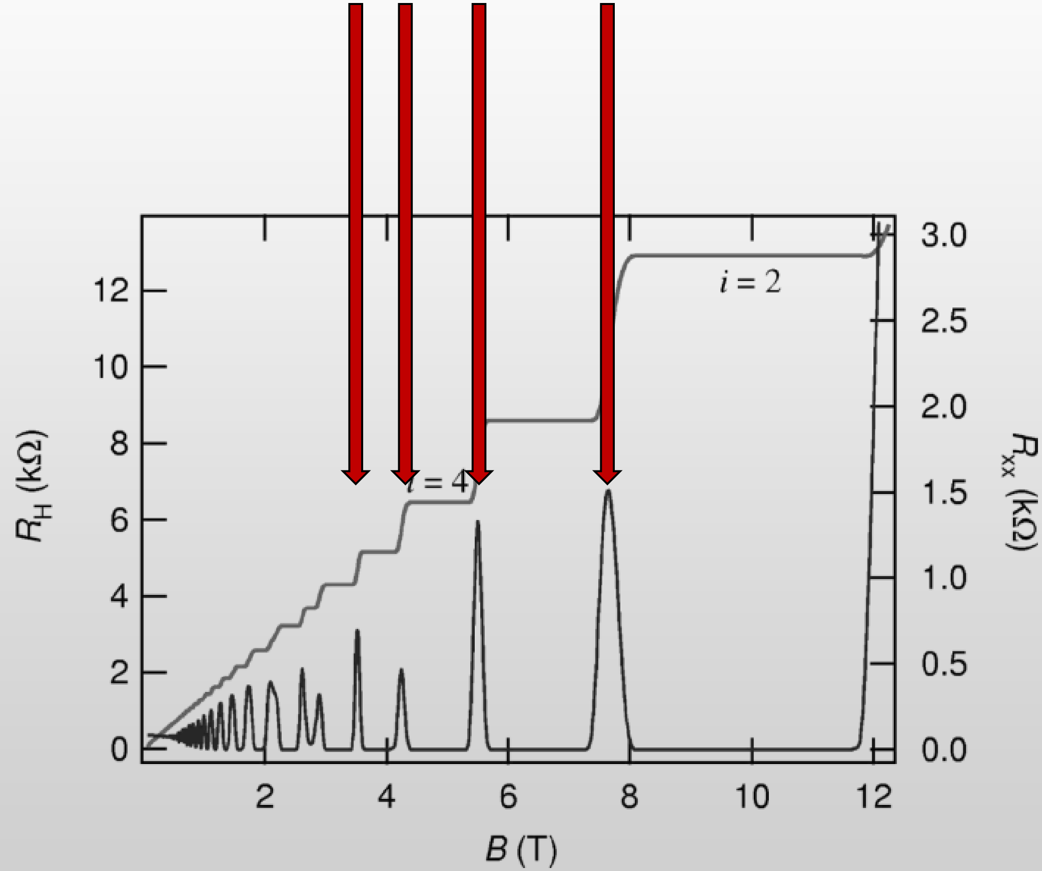


2. **At the quantum phase transition between Hall plateaux**



Anderson Localization is ubiquitous in 1,2-D…Topological exceptions

What does a non-localized, plateau-transition wave function look like?

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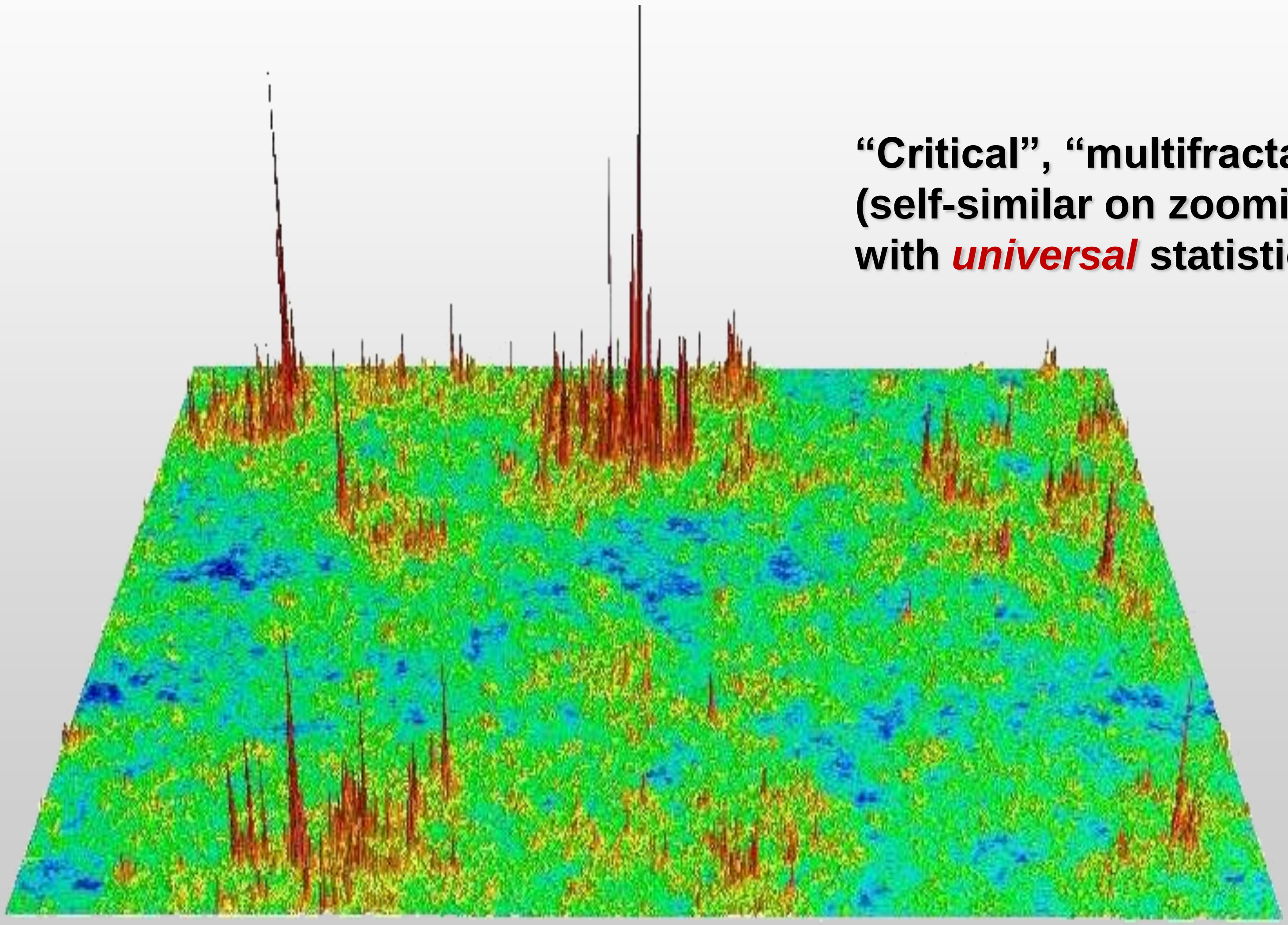

“Critical”, “multifractal” (self-similar on zooming) with *universal* statistics

F. Evers

2D Dirac fractality: …Topology?

- **Where else can Anderson localization be avoided in 2D?**
- **2+1-D Dirac fermions with gauge disorder…but no "natural" realization in graphene, TI surface, etc.**

$$\hat{h} = v_F \left\{ \hat{\sigma}^1 \left[-i\partial_x + A_x(\mathbf{r}) \right] + \hat{\sigma}^2 \left[-i\partial_y + A_y(\mathbf{r}) \right] \right\}$$

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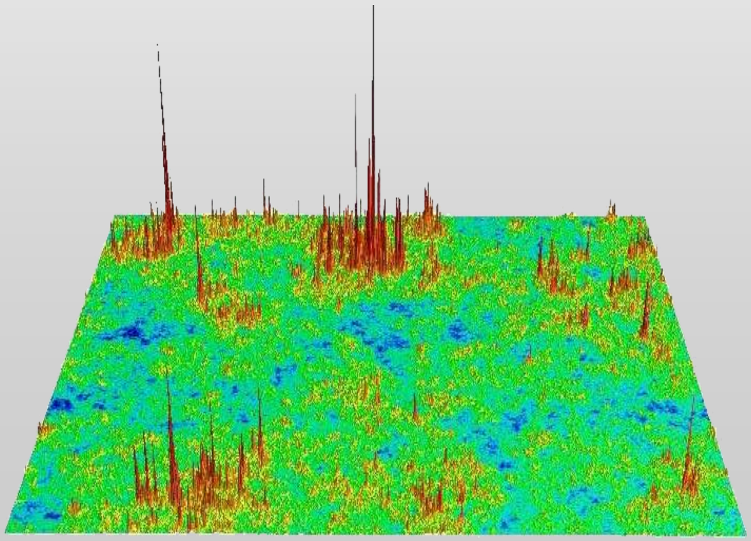
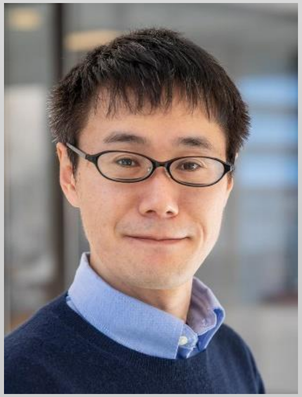
- **"Quenched" QED or QCD: disorder can be abelian or non-abelian (for multiple cones)**
- **Exact solution (conformal field theory): Zero-energy eigenstate is always extended (multifractal)**

Ludwig, Fisher, Shankar, Grinstein (1994)
Nersesyan, Tsvelik, Wenger (1995)
Mudry, Chamon, Wen (1996)
Caux, Kogan, Tsvelik (1996)



3D Topological superconductors: Discovered by Dirac fractality!

From the 2018 film “Annihilation”



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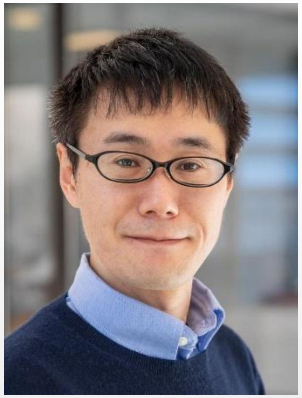

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Schnyder, Ryu, Furusaki, Ludwig (2008):

- **These are surface states of new topological phases in 3D!**
- **Bulk of these new phases: Topological superconductors!**

3D Topological superconductors: Discovered by Dirac fractality!

Class	T	P	S	Spin sym.	$d=2$	$d=3$	Topological realization	Replicated fermion NLσM
C	0	−1	0	SU(2)	$2\mathbb{Z}$	⋯	SQHE (2D $d+id$ TSC)	$\mathrm{Sp}(4n)/\mathrm{U}(2n)$
A (unitary)	0	0	0	U(1)	$\mathbb{Z}$	⋯	IQHE	$\mathrm{U}(2n)/\mathrm{U}(n)\otimes\mathrm{U}(n)$
D	0	+1	0	⋯	$\mathbb{Z}$	⋯	TQHE (2D $p+ip$ TSC)	$\mathrm{O}(2n)/\mathrm{U}(n)$
CI	+1	−1	1	SU(2)	⋯	$2\mathbb{Z}$	3D TSC	$\mathrm{Sp}(4n)\otimes\mathrm{Sp}(4n)/\mathrm{Sp}(4n)$
AIII	0	0	1	U(1)	⋯	$\mathbb{Z}$	3D TSC, chiral TI	$\mathrm{U}(2n)\otimes\mathrm{U}(2n)/\mathrm{U}(2n)$
DIII	−1	+1	1	⋯	$\mathbb{Z}_2$	$\mathbb{Z}$	3D TSC (^{3}He-B)	$\mathrm{O}(2n)\otimes\mathrm{O}(2n)/\mathrm{O}(2n)$
AI (orthogonal)	+1	0	0	SU(2)	⋯	⋯	⋯	$\mathrm{Sp}(4n)/\mathrm{Sp}(2n)\otimes\mathrm{Sp}(2n)$
AII (symplectic)	−1	0	0	⋯	$\mathbb{Z}_2$	$\mathbb{Z}_2$	2D, 3D TIs	$\mathrm{O}(2n)/\mathrm{O}(n)\otimes\mathrm{O}(n)$
BDI	+1	+1	1	SU(2)	⋯	⋯	⋯	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$
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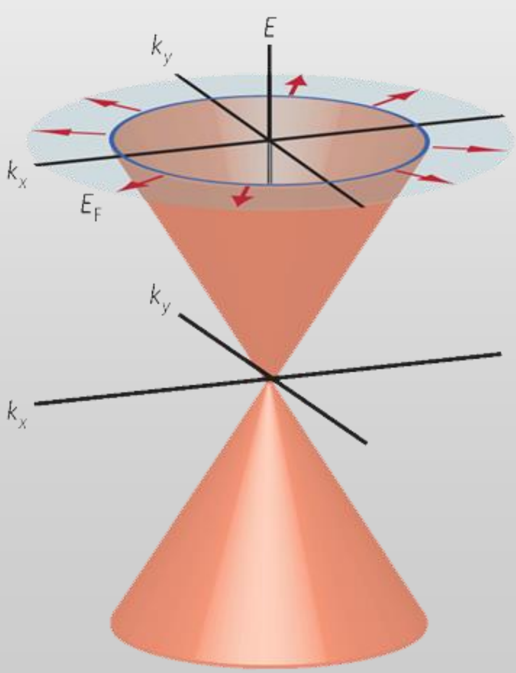
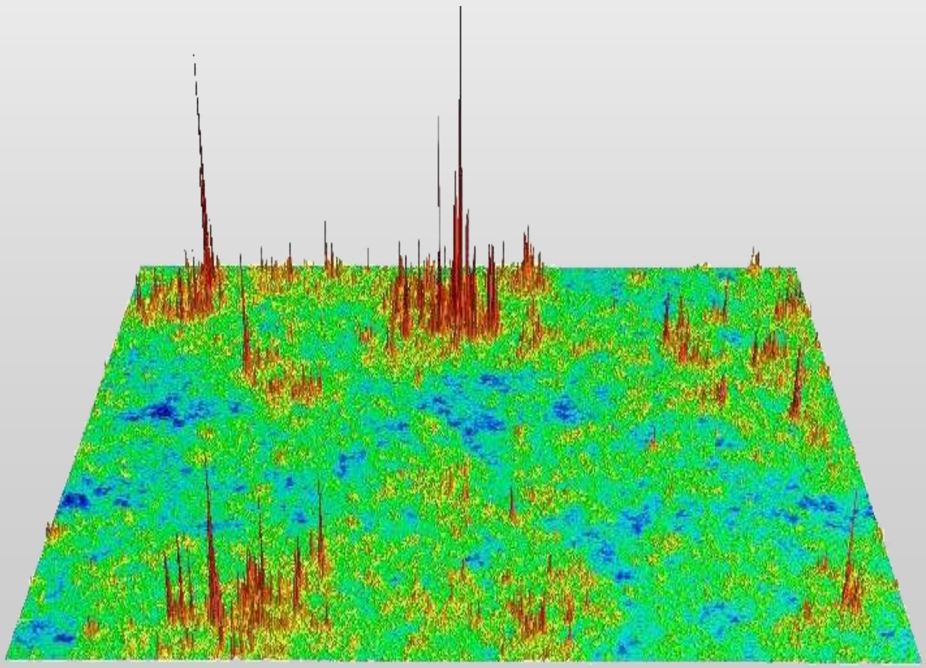
3D Topological superconductors: Discovered by Dirac fractality!

- Zero-energy state is "topologically protected" (not localized)
- …what about all of the others?

$$\hat{h} = v_F \left(-i\hat{\sigma}^1 \left[-i\partial_x + A_x(\mathbf{r})\right] - i\hat{\sigma}^2 \left[-i\partial_y + A_y(\mathbf{r})\right]\right)$$





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3D topological superconductors: Spectrum-wide quantum criticality

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Spectrum-wide quantum criticality conjecture:

Ghorashi, Liao, Foster PRL (2018)

Generic surface states of 3D topological superconductors are critical, statistically identical to plateau transition states in 2D quantum Hall effects

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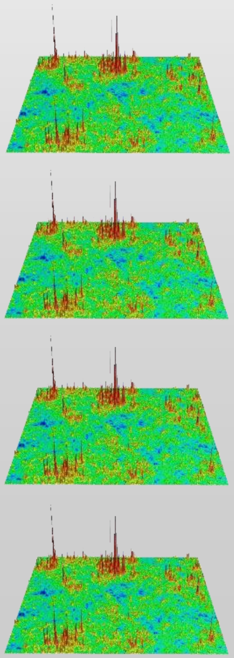


Spectrum-wide quantum criticality conjecture:

Generic surface states of 3D topological superconductors are critical, statistically identical to plateau transition states in 2D quantum Hall effects

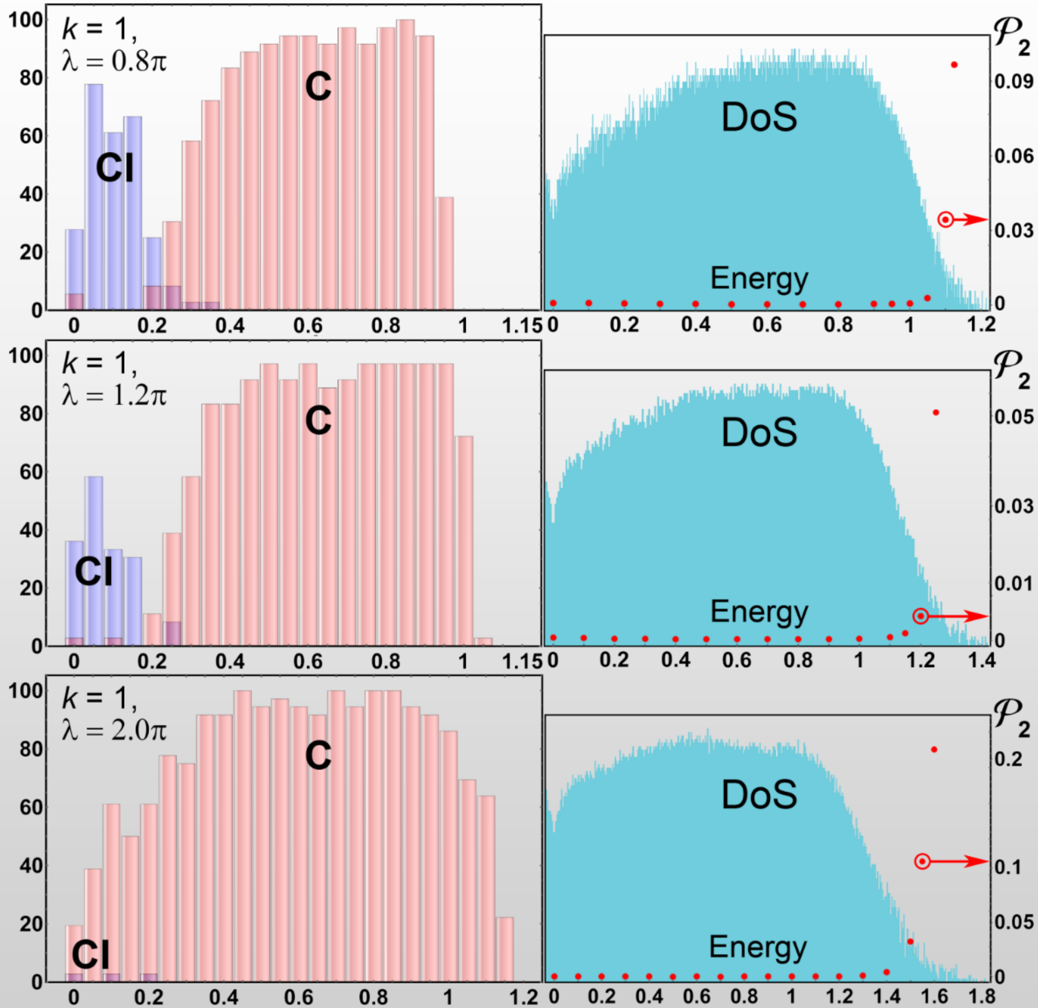
Numerical study: 2D Dirac surface states of 3D TSCs

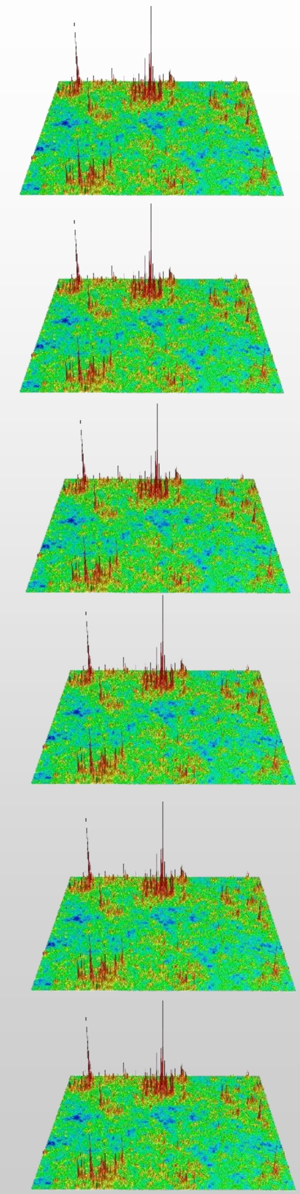
1. Ghorashi, Liao, Foster PRL (2018)
2. Sbierski, Karcher, Foster PRX (2020)
3. Ghorashi, Karcher, Davis, Foster PRB (2020)

Review:
Karcher and Foster
Ann Phys (2021)

Numerical study: 2D Dirac surface states are protected at ALL energies!







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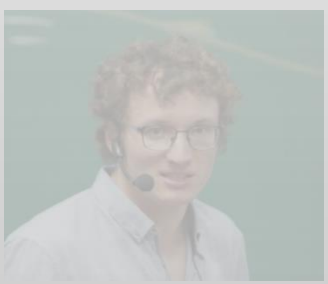

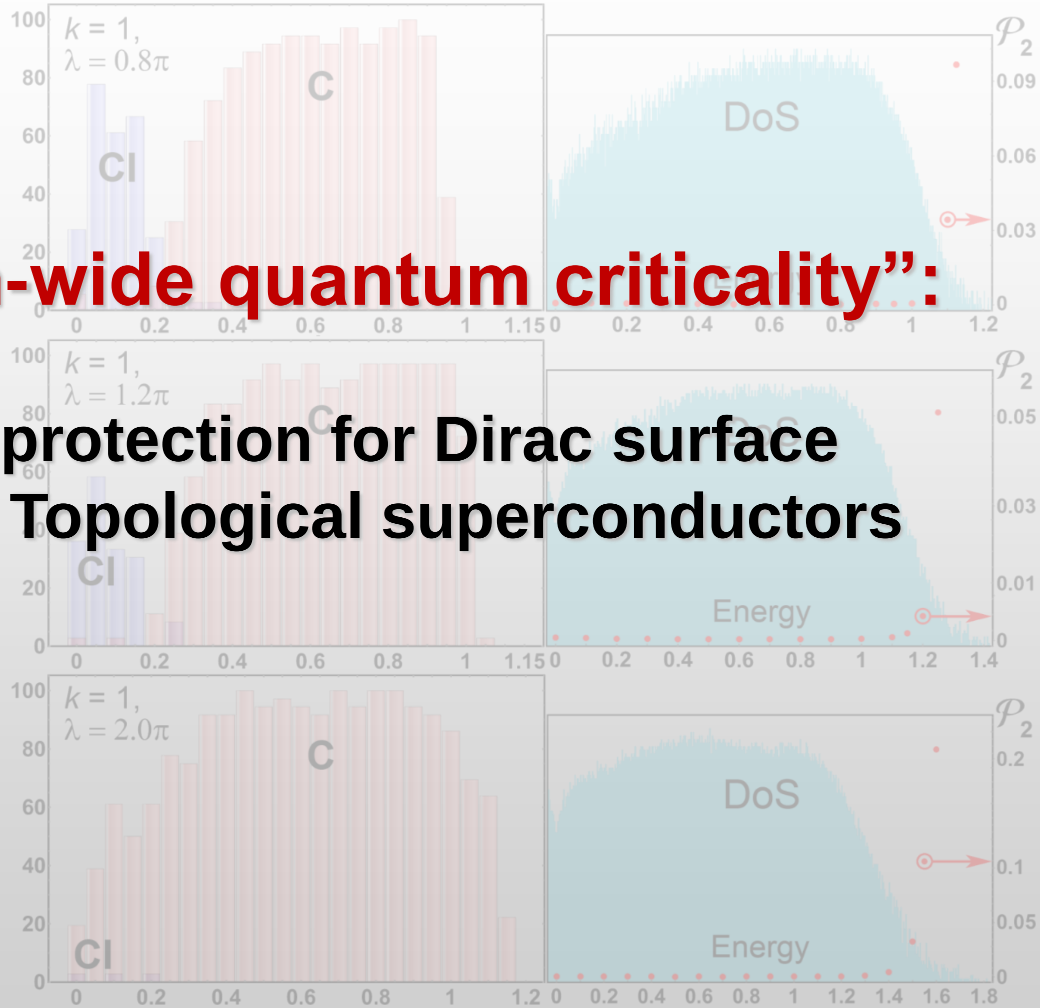

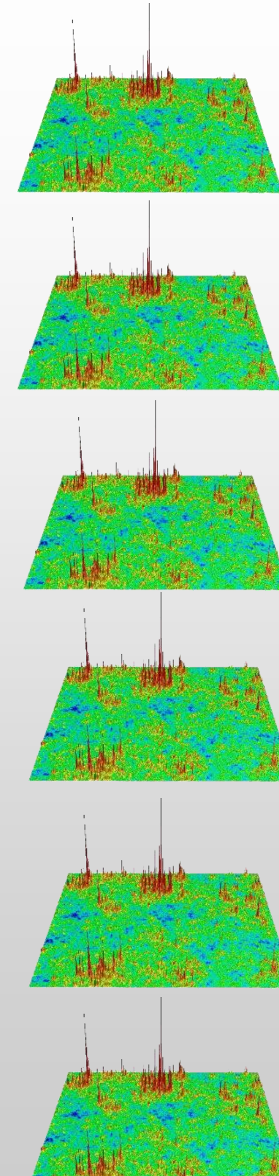

“Spectrum-wide quantum criticality”:

Topological protection for Dirac surface states of 3D Topological superconductors

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Review:
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… Enter the "UFO" (Fragmenting potential)

- We consider a certain lattice model of an AIII topological phase
- There is a strange perturbation that we can add uniformly to the surface (possessing Dirac surface states)

A. Altland, P. Brouwer, J. Dieplinger, M. S. Foster, M. Moreno-Gonzalez, and L. Trifunovic
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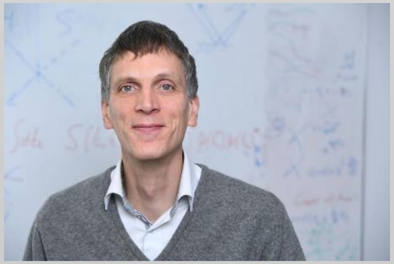

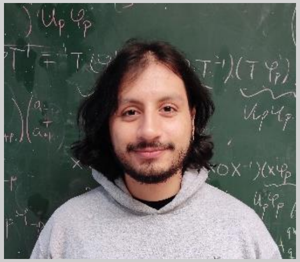


… Enter the "UFO" (Fragmenting potential)

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- There is a strange perturbation that we can add uniformly to the surface (possessing Dirac surface states)

  1. It preserves the defining symmetry of the class

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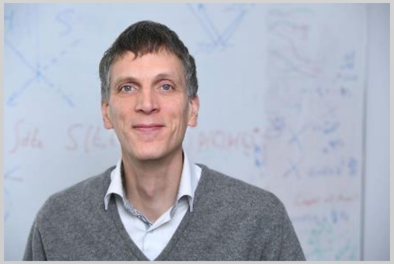
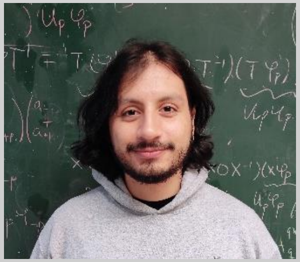





… Enter the "UFO" (Fragmenting potential)

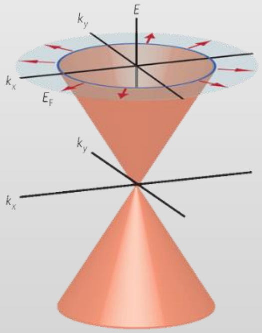
- **We consider a certain lattice model of an AIII topological phase**

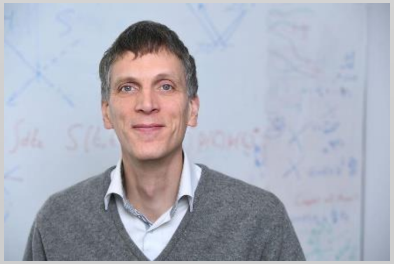
- **There is a strange perturbation that we can add uniformly to the surface (possessing Dirac surface states)**

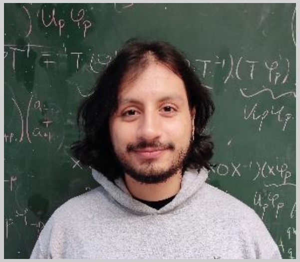
  1. **It preserves the defining symmetry of the class**

  2. **It projects to zero in the Dirac description of the surface**

$$\hat{h} = v_F \left\{ \hat{\boldsymbol{\sigma}} \cdot \left[-i \boldsymbol{\nabla} + \mathbf{A}(\mathbf{r}) \right] \right\}$$



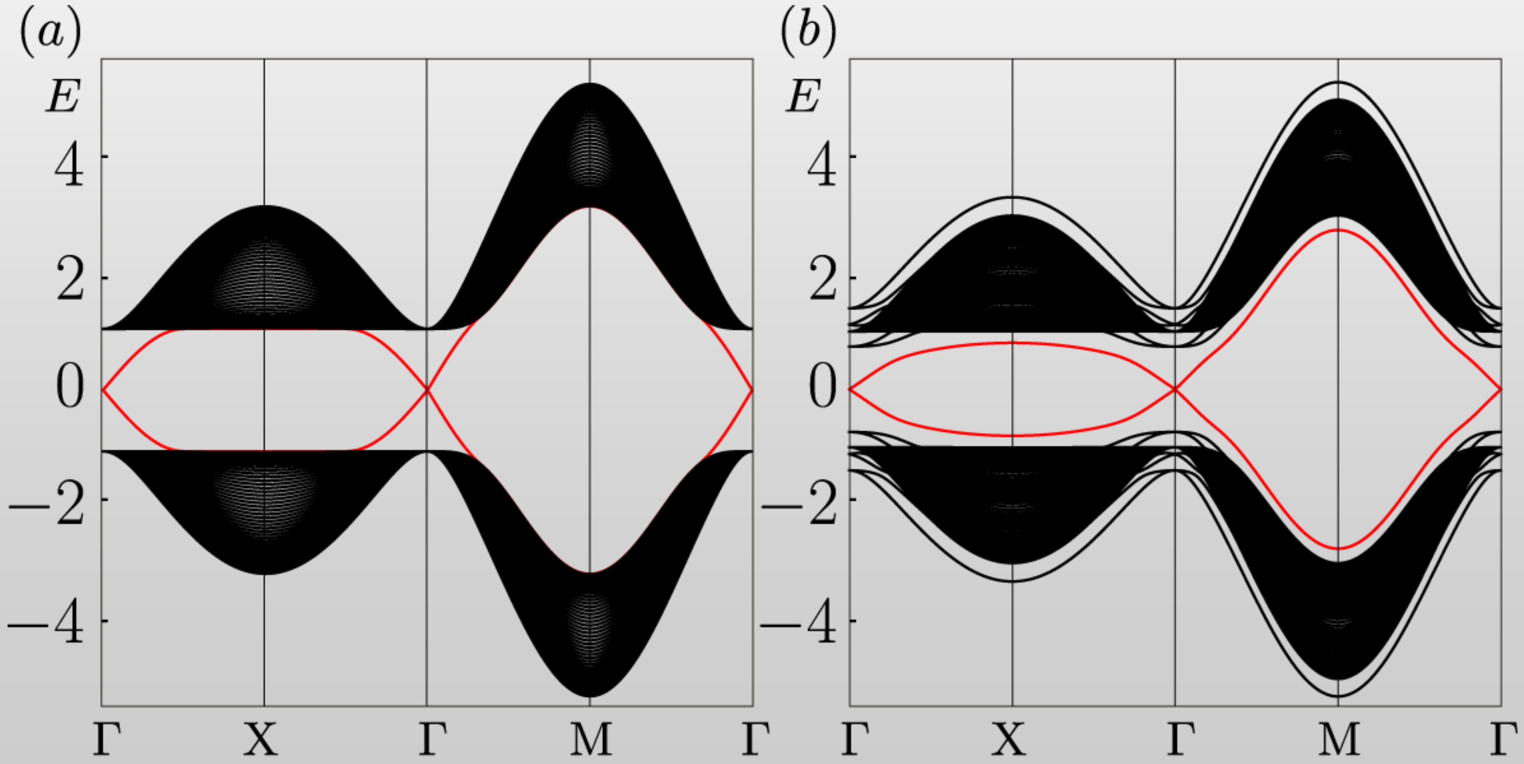




… Enter the "UFO" (Fragmenting potential)

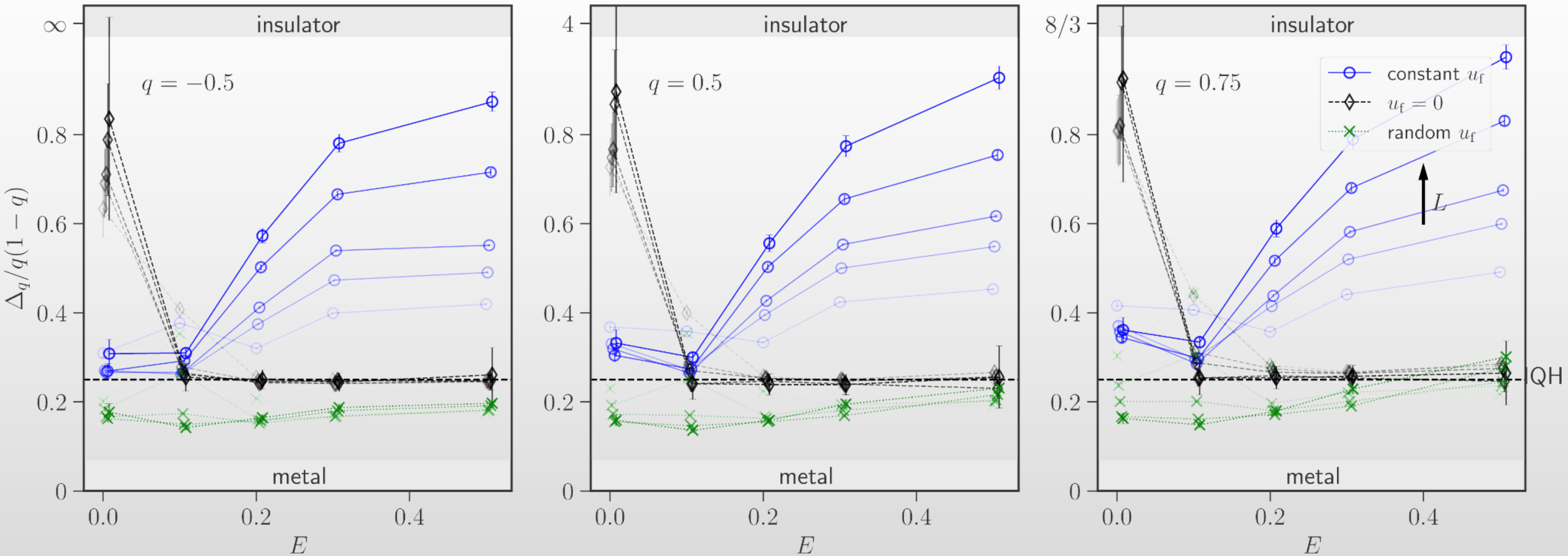
- **We consider a certain lattice model of an AIII topological phase**
- **There is a strange perturbation that we can add uniformly to the surface (possessing Dirac surface states)**

3. …and, it interrupts spectral flow!



… Enter the "UFO" (Fragmenting potential)

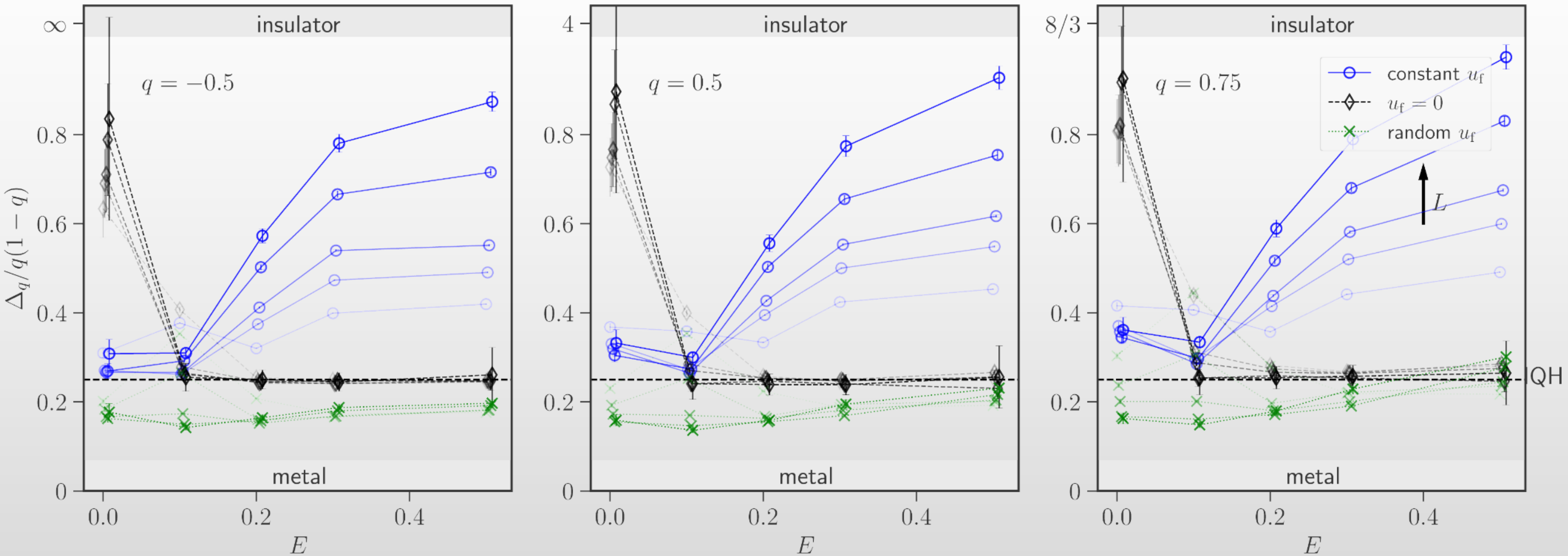
4. …and it Anderson localizes almost all surface states!



- **Black data: No UFO. Spectrum-wide criticality (no localization)**
- **Blue data: Uniform UFO. Localization except at E = 0.**
- **Green data: Random UFO…spectrum-wide critical again!**

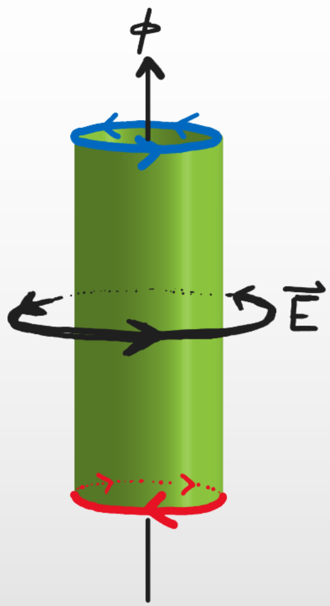
… Enter the "UFO" (Fragmenting potential)

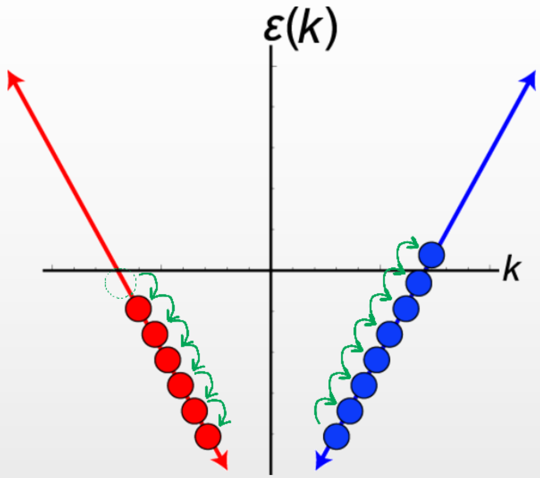
4. …and it Anderson localizes almost all surface states!

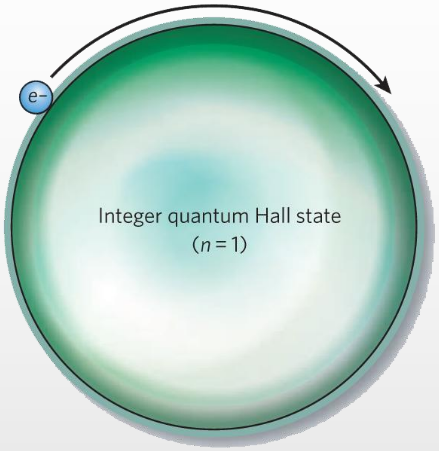


- **All surface states conduct without the UFO**
- **Almost none conduct with it…unless *random* with zero average**
- **Dirac equation can't tell the difference (UFO projects to zero!)**

Why doesn't Dirac work? Missing quantum geometry







Back to Quantum Hall…

- **Edge view: Axial anomaly of 1+1-D Dirac equation**
- **Bulk view: Topological winding number *W* due to Berry curvature**

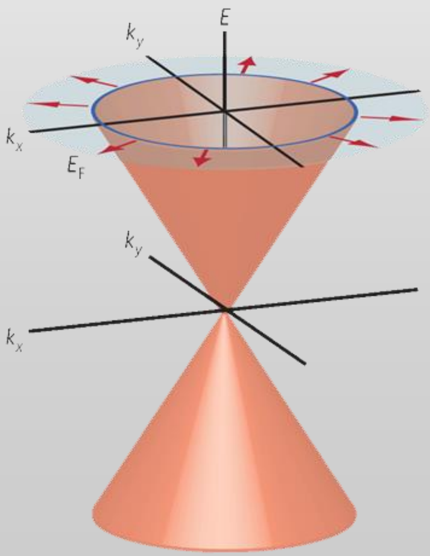
$$\Omega_{\mathbf{k}} = i \left\langle d\alpha_{\mathbf{k}} \right| \wedge \left. d\alpha_{\mathbf{k}} \right\rangle , \qquad W = \frac{1}{2\pi} \int_{\mathbf{k}} \Omega_{\mathbf{k}}$$

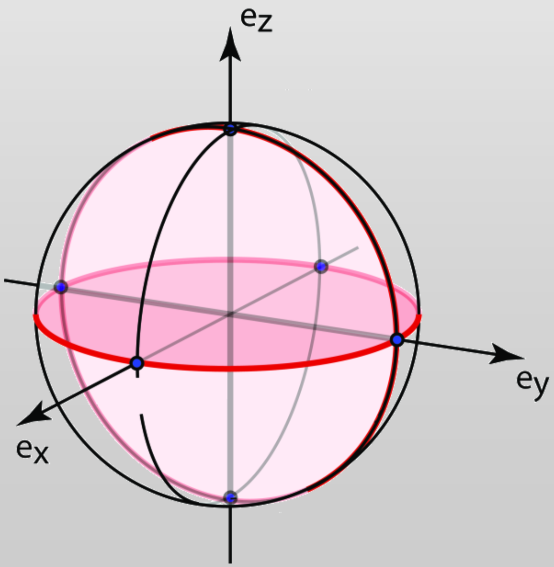
Why doesn't Dirac work? Missing quantum geometry

The "UFO" (fragmenting potential)

- **Induces *surface* Berry curvature "in the sky,"**
  Not captured by the Dirac equation for the surface (Berry-flat)

$$\psi_{\text{Surf}}^{(\text{Dirac})}(k_x, k_y) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \pm\frac{(k_x + ik_y)}{|\mathbf{k}|} \end{bmatrix}$$





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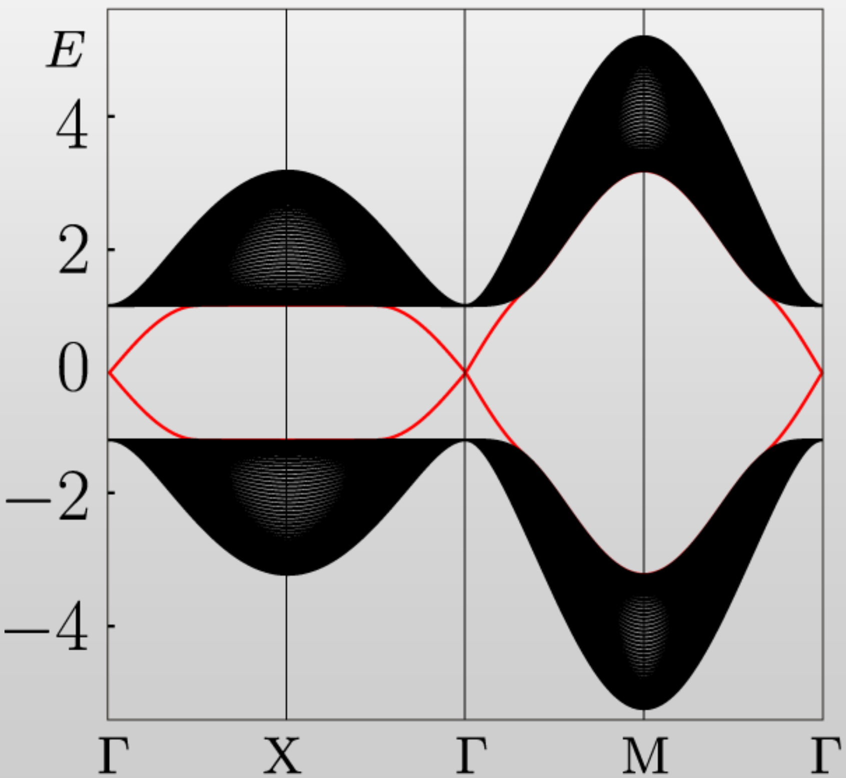

$$\psi^{(\text{Lattice})}_{\text{Surf}}(k_x, k_y) = \begin{bmatrix}\psi_1(\mathbf{k})\\ \psi_2(\mathbf{k})\\ \psi_3(\mathbf{k})\\ \psi_4(\mathbf{k})\end{bmatrix} \rightarrow \begin{bmatrix}1\\ \pm\frac{(k_x+ik_y)}{|\mathbf{k}|}\\ \\ \pm 1\\ \frac{(k_x+ik_y)}{|\mathbf{k}|}\end{bmatrix}$$

No UFO:

4 components deviate from Dirac, but remain Berry-flat

Why doesn't Dirac work? Missing quantum geometry

The "UFO" (fragmenting potential)

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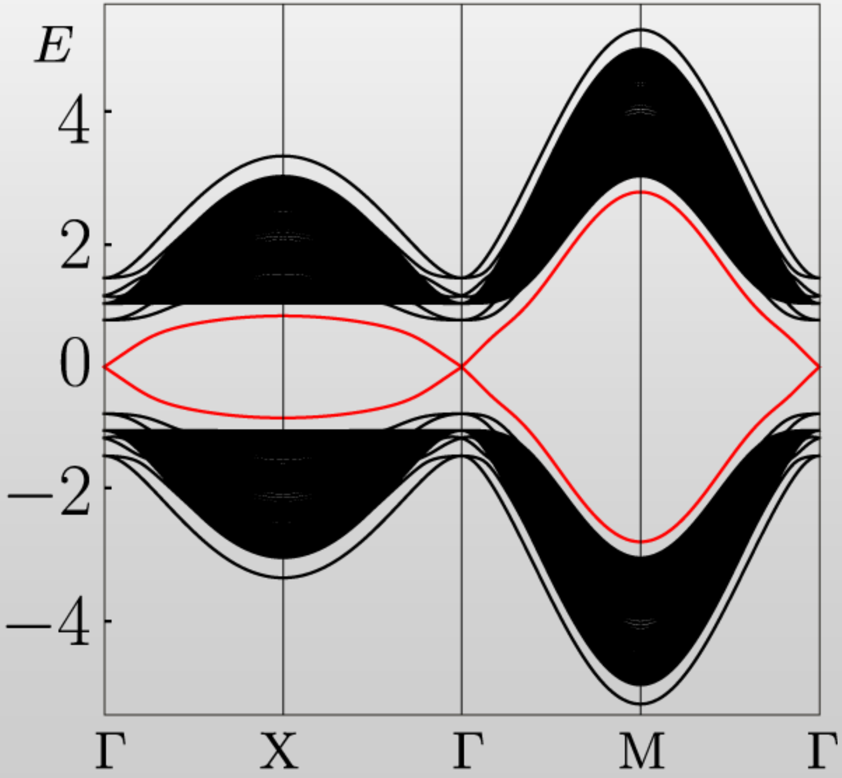
  Not captured by the Dirac equation for the surface (Berry-flat)



$$\psi_{\text{Surf}}^{(\text{Lattice})}(k_x, k_y) = \begin{bmatrix} \psi_1(\mathbf{k}) \\ \psi_2(\mathbf{k}) \\ \psi_3(\mathbf{k}) \\ \psi_4(\mathbf{k}) \end{bmatrix} \to \begin{bmatrix} 1 \\ \pm\frac{(k_x+ik_y)}{|\mathbf{k}|} \\ \pm 1 \\ \frac{(k_x+ik_y)}{|\mathbf{k}|} \end{bmatrix}$$

With UFO:

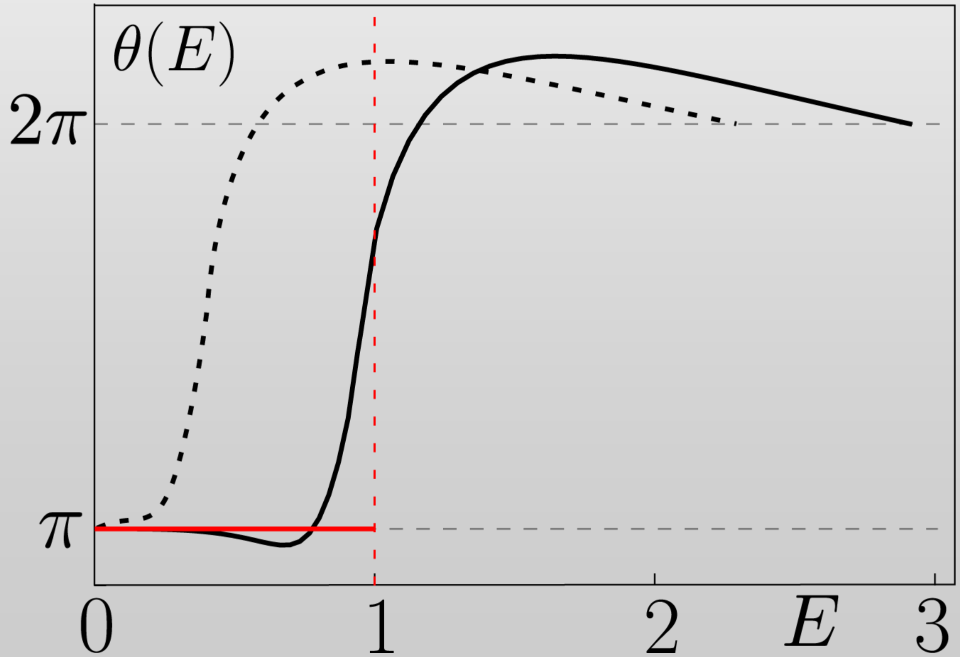
4 components deviate from Dirac,
and develop Berry curvature!

Why doesn't Dirac work? Missing quantum geometry

The "UFO" (fragmenting potential)

- **Induces *surface* Berry curvature "in the sky,"**
  Not captured by the Dirac equation for the surface (Berry-flat)

- **Integrated surface Berry curvature:** $\theta(E) = \pi + \int_{0 \le \varepsilon_{\mathbf{k}} \le E} \Omega_{\mathbf{k}}$



Why doesn't Dirac work? Missing quantum geometry

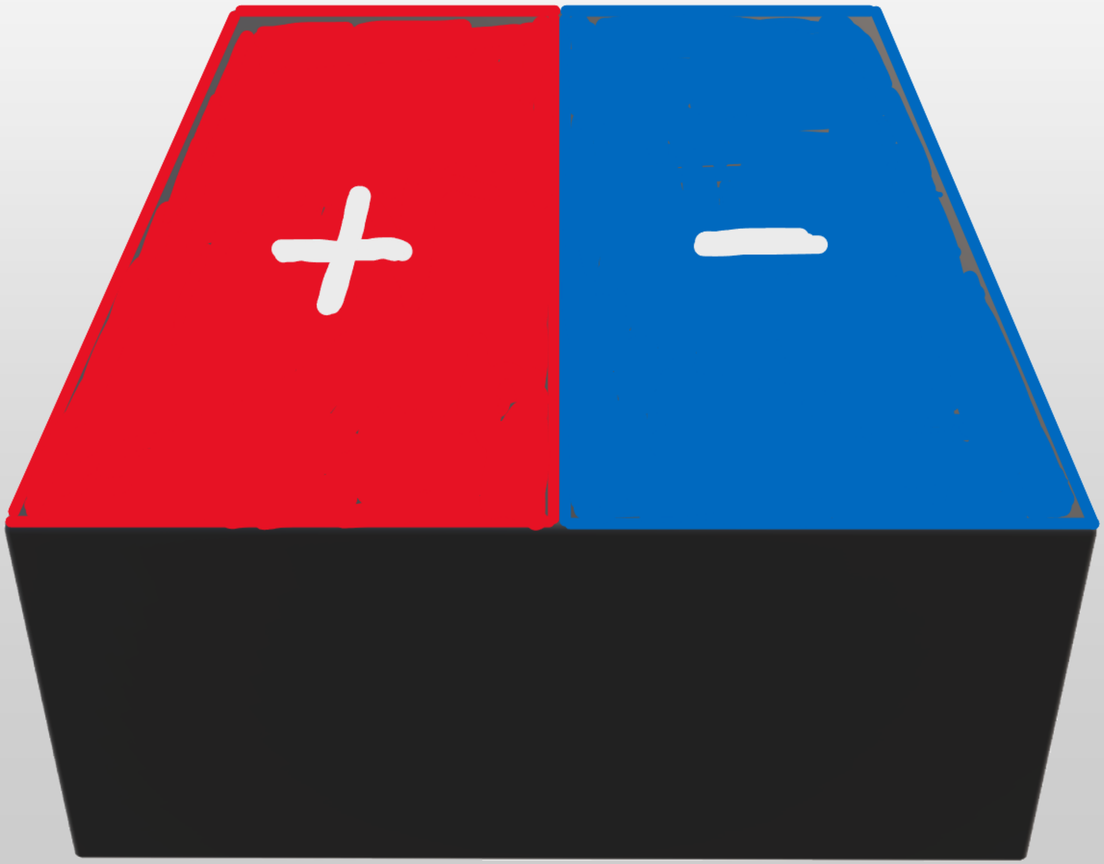
The "UFO" (fragmenting potential)

- Explains when and how spectrum-wide criticality can occur

Why doesn’t Dirac work? Missing quantum geometry

The “UFO” (fragmenting potential)

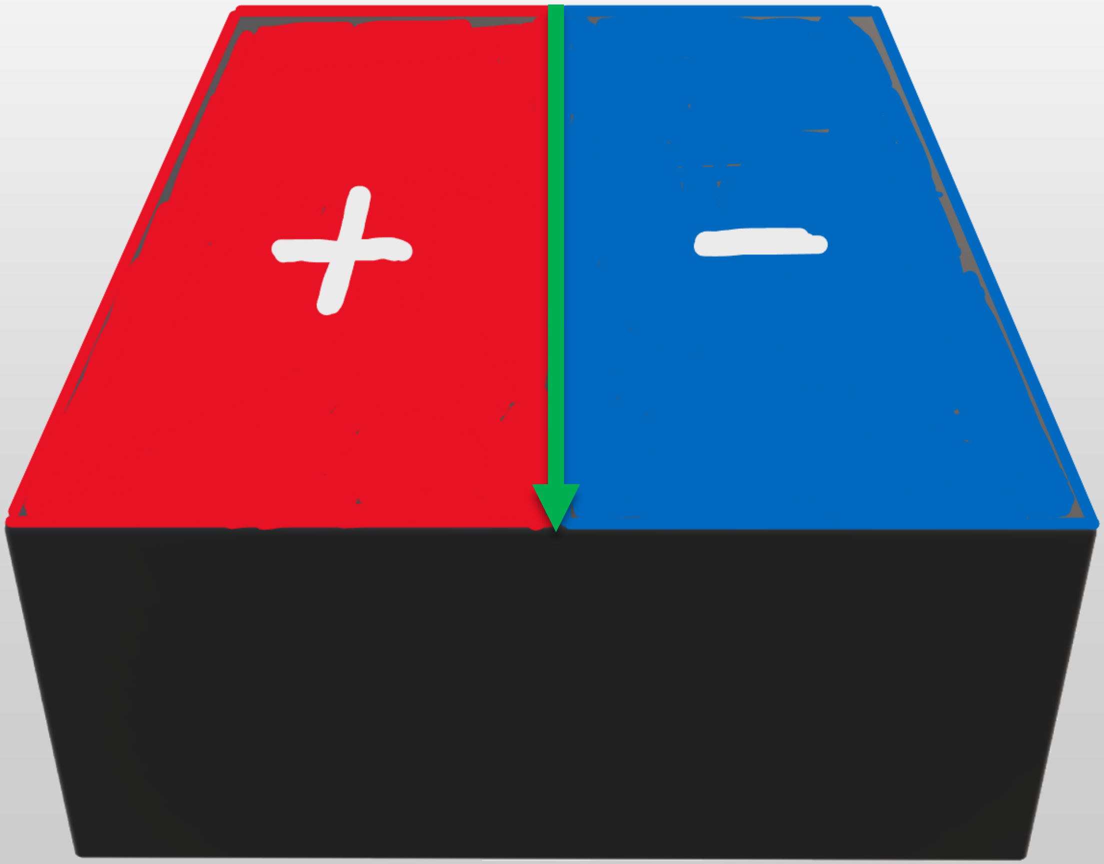
- Explains when and how spectrum-wide criticality can occur



Why doesn't Dirac work? Missing quantum geometry

The "UFO" (fragmenting potential)

- Explains when and how spectrum-wide criticality can occur
- Chiral edge mode "in the sky!"



Why doesn't Dirac work? Missing quantum geometry

The "UFO" (fragmenting potential)

- **Explains when and how spectrum-wide criticality can occur**
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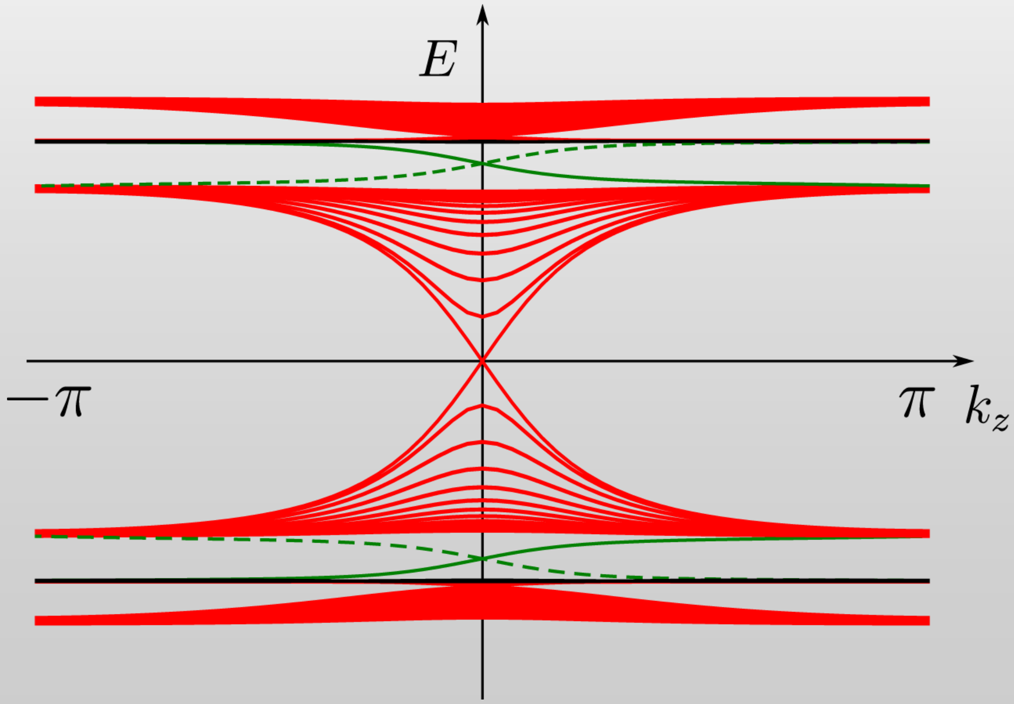
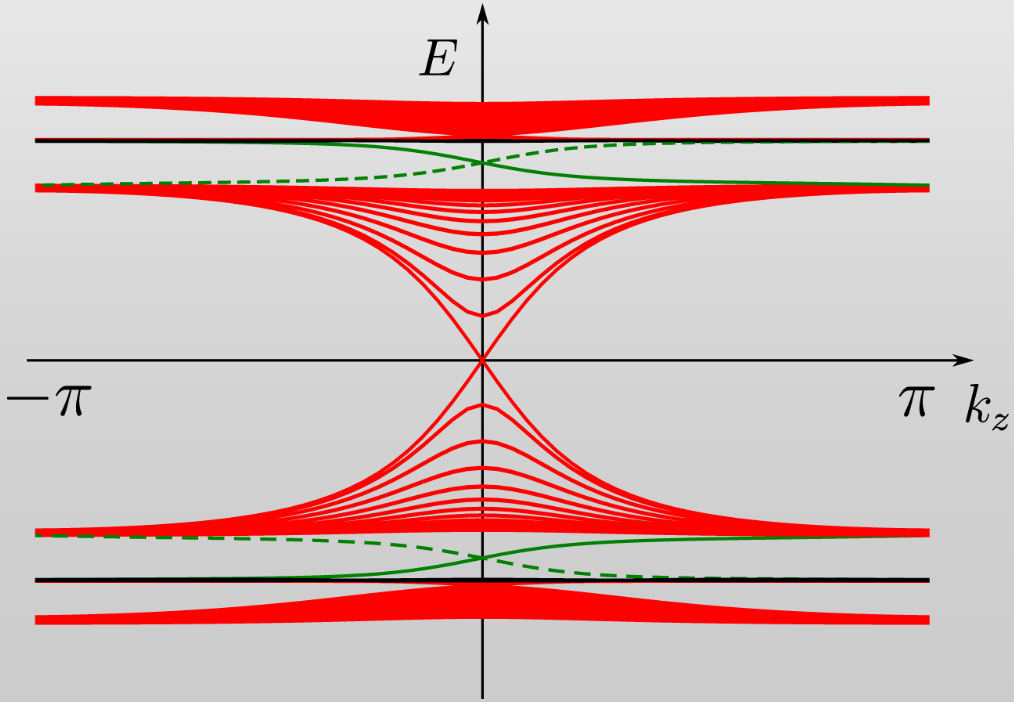


Why doesn't Dirac work? Missing quantum geometry

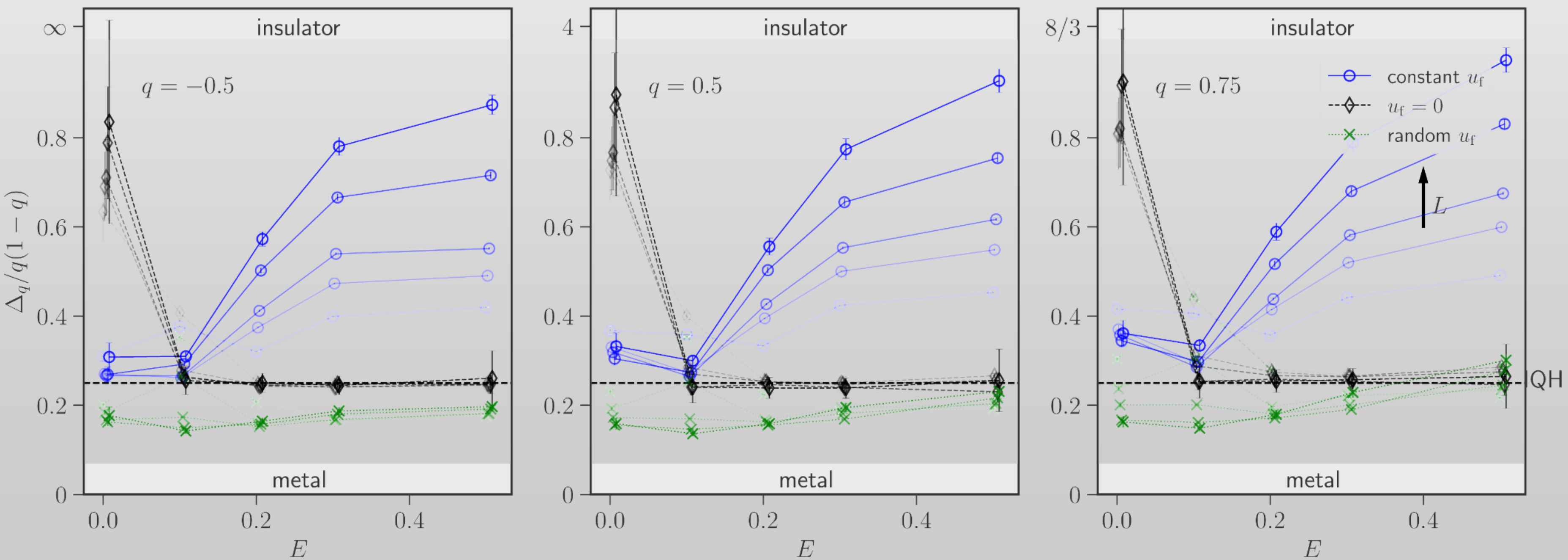
The "UFO" (fragmenting potential)

- **Explains when and how spectrum-wide criticality can occur**
- **Different signs of the potential introduce different surface domains of a "sky-Chern insulator"!**
- **1D Chiral edge modes form at boundaries**



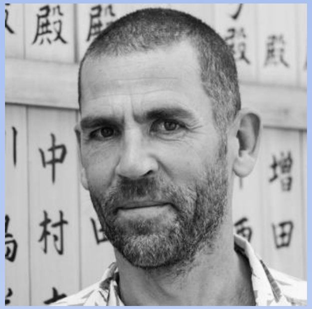
Why doesn't Dirac work? Missing quantum geometry

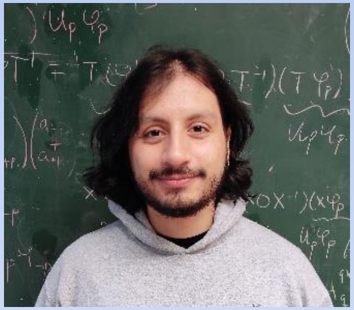
- **Different signs of the potential introduce different surface domains of a "sky-Chern insulator"!**
- **1D Chiral edge modes form at boundaries**
- **Spectrum-wide quantum criticality: When these edge modes percolate! (Random UFO with zero average…)**



Summary: Fragility of spectral flow in topological insulators

class	$d=1$	$d=2$	$d=3$
A	0	$\mathbb{Z}^{\checkmark}$	0
AIII	$\mathbb{Z}^{\times}$	0	$\mathbb{Z}^{\times}$
AI	0	0	0
BDI	$\mathbb{Z}^{\times}$	0	0
D	$\mathbb{Z}_2^{\times}$	$\mathbb{Z}^{\checkmark}$	0
DIII	$\mathbb{Z}_2^{\times}$	$\mathbb{Z}_2^{\checkmark}$	$\mathbb{Z}^{\checkmark / \times}$
AII	0	$\mathbb{Z}_2^{\checkmark}$	$\mathbb{Z}_2^{\checkmark}$
CII	$2\mathbb{Z}^{\times}$	0	$\mathbb{Z}_2^{\times}$
C	0	$2\mathbb{Z}^{\checkmark}$	0
CI	0	0	$2\mathbb{Z}^{\times}$




✓: Not localizable

✗ : Localizable (e.g, UFO in 3D)

A. Altland, P. Brouwer, J. Dieplinger, M. S. Foster, M. Moreno-Gonzalez, and L. Trifunovic **arXiv:2308.12931**